Limiting Distribution of Frobenius Numbers for $n = 3$

V.Shur* Ya. Sinai† A. Ustinov‡

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1 Introduction

The purpose of this paper is to give a complete derivation of the limiting distribution of large Frobenius numbers outlined in [1] and fill some gaps formulated there as hypotheses. We start with the basic definitions and descriptions of some results.

Consider n mutually coprime positive integers $a_1, a_2, \ldots, a_n$. This means that there is no $r > 1$ such that each a_j, $1 \leqslant j \leqslant n$, is divisible by r. Take N which later will tend to infinity and will be our main large parameter. Introduce the ensemble Q_N of mutually coprime $a = (a_1, \ldots, a_n)$, $1 \leqslant a_j \leqslant N$, $1 \leqslant j \leqslant n$ and P_N be the uniform probability distribution on Q_N. For each $a \in Q_N$ denote by $F(a)$ the largest integer number that is not representable in the form $x = x_1a_1 + \cdots + x_na_n$, where x_j are non-negative integers. $F(a)$ can be considered as a random variable defined on Q_N. The basic problem which will be discussed in this paper is the existence and the form of the limiting distribution for the normalized Frobenius numbers $f(a) = \dfrac{1}{N^{1+\frac{1}{n-1}}}F(a)$. The reason for this normalization will be explained below.

The case of $n = 2$ is simple in view of the classical result of Sylvester (see [7]) according to which $F(a_1, a_2) = a_1a_2 - a_1 - a_2$. It shows that in a typical situation F grow as N^2. The first non-trivial case is $n = 3$ where $F(a)$ grow as $N^{3/2}$ It is known (see [10]) that the numbers $F(a_1, a_2, a_3)$ have weak asymptotics:

$$\frac{1}{x_1x_2a_3^{7/2}} \sum_{a_1 \leqslant x_1a_3} \sum_{a_2 \leqslant x_2a_3} \left(F(a_1, a_2, a_3) - \frac{8}{\pi}\sqrt{a_1a_2a_3} \right) = O_{x_1,x_2,\varepsilon}\left(a_3^{-1/6+\varepsilon} \right)$$

For arbitrary n the only result known to us is the following theorem proven in [1].

*Mathematics Department, Moscow State University, Russia, vladimir@chg.ru
†Mathematics Department, Princeton University, USA, sinai@math.princeton.edu
‡Khabarovsk Division of Institute for Applied Mathematics, Far Eastern Branch of the Russian Academy of Science, Russia, ustinov@iam.khv.ru

Theorem 1. *Under some additional technical condition (see [1]) the family of probability distributions of* $f_N(a) = \frac{1}{N^{1+\frac{1}{n-1}}} F(a)$ *is weakly compact. This means that for every* $\varepsilon > 0$ *one can find* $\mathcal{D} = \mathcal{D}(\varepsilon)$ *such that*

$$P_N \left\{ \frac{1}{N^{1+\frac{1}{n-1}}} F(a) \leqslant \mathcal{D} \right\} \geqslant 1 - \varepsilon.$$

In this theorem $\varepsilon, \mathcal{D}$ do not depend on N. It also implies the existence of the limiting points (in the sense of weak convergence) for the sequence of probability distributions of $f_N(a)$. As was already mentioned, in this paper we shall study the limiting distribution of $f_N(a) = \frac{1}{N^{3/2}} F(a)$, $a = (a_1, a_2, a_3)$ as $N \to \infty$. This distribution is not universal and will be described below.

Take any ρ, $0 < \rho < 1$, and consider its expansion into continued fraction

$$\rho = [h_1, h_2, \dots, h_s, \dots] \tag{1}$$

where $h_j \geqslant 1$ are integers. If ρ is rational then the continued fraction (1) is finite. The finite continued fractions $\rho = [h_1, \dots, h_s] = \dfrac{p_s}{q_s}$ are called the s-approximants of ρ. The numbers q_s satisfy the recurrent relations

$$q_s = h_s q_{s-1} + q_{s-2}, \quad s \geqslant 2 \tag{2}$$

Introduce the Gauss measure on $[0, 1]$ given by the density $\pi(x) = \frac{1}{\ln 2(1+x)}$. Then the elements of the continued fraction (1) become random variables. It is well-known that their probability distributions are stationary in the sense that the distributions of any $h_{m-k}, h_{m-k+1}, \dots, h_m \dots, h_{m+k}$ do not depend on m. We shall need the values of $s = s_1$, such that q_{s_1} is the first q_s greater than $\sqrt{N}$. It was proven in [5] that $q_{s_1}/\sqrt{N}$ have a limiting distribution. More precisely, the following theorem is true.

Theorem 2. *Let* k *be fixed and* $s(R)$ *be the smallest* s *for which* $q_s \geqslant R$. *As* $R \to \infty$ *there exists the joint limiting probability distribution of* $\frac{q_{s(R)}}{R}$, $h_{s(R)-k}$, $\dots$, $h_{s(R)+k}$.

In the paper [11] the analytic form of this distribution was given.

Consider the subensemble $Q_N^{(0)} \subset Q_N$ for which a_1, a_3 are coprime. Then there exists $a_1^{-1} (\mathrm{mod}\ a_3)$, $1 \leqslant a_1^{-1} < a_3$. Denote $\rho = \frac{a_1^{-1} a_2}{a_3}$. The expansion of ρ into continued fraction will be needed below. Clearly, ρ is a rational number. However, the following theorem is valid.

Theorem 3. *As before, consider* s_1 *such that* $q_{s_1 - 1} < \sqrt{N} < q_{s_1}$. *Then in the ensemble* $Q_N^{(0)}$ *equipped with the uniform measure, for any* $k > 0$ *and* $N \to \infty$ *there exists the joint limiting probability distributions of* $\frac{q_{s_1}}{\sqrt{N}}$, h_{s_1-k}, $\dots$, h_{s_1+k} *which coincides with the distribution in theorem 2.*

A stronger version of theorem 3 is also valid.

Theorem 4. *Let the first elements of the continued fraction for ρ be fixed: $h_1, h_2, \dots, h_l$. Then under this condition and as $N \to \infty$ the conditional distributions of $\frac{q_{s_1}}{\sqrt{N}}$, h_{s_1-k}, $\dots$, h_{s_1+k} converge to the same limit as in theorems 2 and 3.*

All these theorems will be proven in section 3. Now we can formulate the main result of this paper.

Theorem 5. *There exists the limiting distribution of $f_N(a) = f_N((a_1, a_2, a_3))$, $(a_1, a_2, a_3) \in Q_N$ as $N \to \infty$.*

The proof of the main theorem is given in section 2. First we consider the ensemble $Q_N^{(0)}$ and then explain how to extend the proof to Q_N.

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2 The limiting Distribution of $f_N(a)$.

Return back to the case of arbitrary n. Introduce arithmetic progressions

$$\Pi_r = \{r + m a_n, m \geqslant 0\}, \quad 0 \leqslant r < a_n.$$

For non-negative integers $x_1, \dots, x_{n-1}$ such that $x_1a_1 + x_2a_2 + \cdots + x_{n-1}a_{n-1} \in \Pi_r$ we write

$$x_1a_1 + \cdots + x_{n-1}a_{n-1} = r + m(x_1, \dots, x_{n-1})a_n.$$

Define $\overline{m}(r) = \min\limits_{x_1 \dots, x_{n-1}} m(x_1, \dots, x_{n-1})$ and put

$$F_1(a) = \max_{0 \leqslant r < a_n} \min_{\substack{x_1, \dots, x_{n-1} \\ x_1a_1 + \dots + x_{n-1}a_{n-1} \in \Pi_r}} (r + m(x_1, \dots, x_{n-1})a_n) =$$

$$= \max_{0 \leqslant r < a_n} \min_{x_1a_1 + \ldots + x_n + a_{n-1} \equiv r (\operatorname{mod} a_n)} (x_1a_1 + \ldots + a_{n-1}a_{n-1})\,.$$

It was proven in [3] that $F(a) = F_1(a) - a_n$. A slightly weaker statement can be found in [1]. Since in a typical situation a_j grow as N while $F_1(a)$ grow as $N^{1+\frac{1}{n-1}}$ (see also [1]) the limiting behavior of $\frac{F(a)}{N^{1+\frac{1}{n-1}}}$ and $\frac{F_1(a)}{N^{1+\frac{1}{n-1}}}$ is the same, but the analysis of $\frac{F_1(a)}{N^{1+\frac{1}{n-1}}}$ is slightly simpler. Let us write for $n = 3$

$$x_1a_1 + x_2a_2 = r + m(x_1, x_2)a_3$$

or

$$x_1a_1 + x_2a_2 \equiv r (\operatorname{mod} a_3) \tag{3}$$

We assume that a_1, a_3 and a_2, a_3 are coprime. Therefore there exists a_1^{-1}, $1 \leqslant a_1^{-1} < a_3$, such that $a_1 \cdot a_1^{-1} \equiv 1 (\bmod\, a_3)$. Choose a_1^{-1} so that $1 \leqslant a_1^{-1} < a_3$ and rewrite (3) as follows

$$x_1 + a_{12} x_2 \equiv r_1 (\bmod\, a_3) \tag{4}$$

where $a_{12} \equiv a_1^{-1} a_2 (\bmod\, a_3)$, $0 < a_{12} < a_3$ and $r_1 \equiv r a_1^{-1} (\bmod\, a_3)$, $0 \leqslant r_1 < a_3$. From (4)

$$a_{12} x_2 \equiv (r_1 - x_1)(\bmod\, a_3) \tag{5}$$

The expression (5) has a nice geometric interpretation. Consider $S = [0, 1, \ldots, a_3 - 1]$ as a "discrete circle". Let $\mathcal{R}$ be the rotation of this circle by a_{12}, i.e. $\mathcal{R}x = x + a_{12} (\mathrm{mod} a_3)$. Then $\mathcal{R}^p x = x + p a_{12} (\mathrm{mod} a_3)$ and (5) means that $r_1 - x_1$ belongs to the orbit of 0 under the action of $\mathcal{R}$. From the definition of $F_1(a)$

$$F_1(a) = \max_{0 \leqslant r < a_3} \min_{\substack{x_1 a_1 + x_2 a_2 \equiv r (\bmod\, a_3) \\ 0 \leqslant x_1, x_2 < a_3}} (x_1 a_1 + x_2 a_2) =$$

$$= N^{3/2} \max_{0 \leqslant r_1 < a_3} \min_{x_1 + x_2 a_{12} \equiv r_1 \quad (\bmod\, a_3)} \left(\frac{x_1}{\sqrt{N}} \frac{a_1}{N} + \frac{x_2}{\sqrt{N}} \frac{a_2}{N} \right) \tag{6}$$

Choose $h^{(j)} = (h_1^{(j)}, \ldots, h_m^{(j)})$, $j = 1, 2, 3$ and denote by $Q^{(0)}_{N, h^{(1)}, h^{(2)}, h^{(3)}}$ the ensemble of $a = (a_1, a_2, a_3) \in Q_N^{(0)}$ such that the first m elements of the continued fractions of $\frac{a_j}{N}$ are given by $h^{(j)}$, $j = 1, 2, 3$. This step means the localization of the ensemble $Q_N^{(0)}$. It is easy to see that for every $\varepsilon > 0$ one can find rational α_1, α_2, α_3 and m such that $\left| \frac{a_j}{N} - \alpha_j \right| \leqslant \varepsilon$, $1 \leqslant j \leqslant 3$. Then in (6) one can replace $\frac{a_j}{N}$ by α_j. Since $\frac{x_j}{\sqrt{N}}$ will take the values $O(1)$ the whole expression in (6) takes values $O(1)$ and instead of (6) we may consider

$$\max_{r_1} \min_{x_1 + a_{12} x_2 \equiv r_1 \quad (\bmod\, a_3)} \left(\frac{x_1}{\sqrt{N}} \alpha_1 + \frac{x_2}{\sqrt{N}} \alpha_2 \right) \tag{7}$$

with the error $O(\varepsilon)$. We assume that in the ensemble $Q^{(0)}_{N, h^{(1)}, h^{(2)}, h^{(3)}}$ we also have the uniform distribution.

We shall need some facts from the theory of rotations of the circle. According to our assumption a_{12} and a_3 are coprime. Therefore $\mathcal{R}$ is ergodic in the sense that $\mathcal{R}^{a_3} = Id$ and a_3 is the smallest number with this property. Put $\rho = \frac{a_{12}}{a_3}$ and write down the expansion of ρ into continued fraction: $\rho = [h_1, h_2, \ldots, h_{s_0}]$. Let also be $\rho_s = [h_1, h_2, \ldots, h_s] = \frac{p_s}{q_s}$ and s_1 is such that $q_{s_1 - 1} < \sqrt{N} < q_{s_1}$.

It will be more convenient to consider the usual unit circle instead of S and use the same letter $\mathcal{R}$ for the rotation of the unit circle by ρ. Introduce the interval $\Delta_0^{(p)}$ bounded by 0 and $\{q_p \rho\}$ and $\Delta_j^{(p)} = \mathcal{R}^j \Delta_0^{(p)}$. Using the induction one can show that $\Delta_j^{(p)}$, $0 \leqslant j < q_{p+1}$ and $\Delta_j^{(p+1)}$, $0 \leqslant j' < q_p$ are pair-wise disjoint and their union is the whole circle except the boundary points (see [5]). Denote by $\eta^{(p)}$ the partition of the unit circle into $\Delta_j^{(p)}$, $\Delta_{j'}^{(p+1)}$. Then $\eta^{(p+1)} \geqslant \eta^{(p)}$ in the sense that each element of $\eta^{(p)}$ consists of several elements of $\eta^{(p+1)}$. More precisely, $\Delta_0^{(p-1)}$ consists of h_p elements $\Delta_j^{(p)}$ and one elements $\Delta_0^{(p+1)}$. The partitions $\eta^{(p)}$ show how the orbit of 0 fills the circle.

Return back to the discrete circle S. The partitions $\eta^{(p)}$ can be constructed in the same way as in the continuous case. We have to analyze

$$\max_{0\leqslant r_1<a_3} \min_{\substack{x_1,x_2\\ x_1+a_{12}x_2\equiv r_1(\bmod a_3)}} \left(\frac{x_1}{\sqrt{N}}\alpha_1+\frac{x_2}{\sqrt{N}}\alpha_2\right) \tag{8}$$

for given $\alpha_1,\alpha_2,\ 0<\alpha_1,\alpha_2<1$.

Lemma 1. *There exists some number $C_1(\alpha_1,\alpha_2)=C_1$ such that for any r_1 the point x_1 giving $\min\left(\frac{x_1}{\sqrt{N}}\,\alpha_1+\frac{x_2}{\sqrt{N}}\,\alpha_2\right)$ under the condition is such that r_1-x_1 ($x_1+a_{12}x_2\equiv r_1 \pmod{a_3}$) is an end-point of some element of the partition $\eta^{(s_1+m_1)}$. Here $m_1\geq 0$ is such that $qs_1+m_1/q_{s_1}\ \leq\ C_1(\alpha_1,\alpha_2)$*

The proof is simple. In any case r_1-x_1 is an end-point of some element of the partition $\eta^{(s_1+m_1)}$. If m_1 is too big then $\frac{x_2}{\sqrt{N}}$ is too big because it takes too much time to reach an end-point of $\eta^{(s_1+m_1)}$ which is not an end-point of one of the previous partitions. We can choose y_1 so that r_1-y_1 will be an end-point of some element of $\eta^{(s_1)}$ and the linear combination $\frac{y_1}{\sqrt{N}}\alpha_1+\frac{y_2}{\sqrt{N}}\alpha_2$ is smaller. This completes the proof of the lemma.

Its meaning is the following. If r_1-x_1 is an end-point of $\eta^{(s_1+m_1)}$ with too big m_1 then x_2 will be also too big.

Lemma 2 shows that x_1 also cannot be too big.

Lemma 2. *There exists an integer $m_2>0$ depending on α_1,α_2 the ratio q_{s_1}/N and the elements of the continued fraction $h_{s_1},h_{s_1+1},\dots,h_{s_1+m_2}$ of ρ such that for any r_1 the interval $[r_1-x_1,r_1]$ corresponding to the minimum of*

$$\frac{x_1}{\sqrt{N}}\,\alpha_1\;+\;\frac{x_2}{\sqrt{N}}\,\alpha_2$$

has not more than m_2 elements of $\eta^{(s_1)}$.

The proof is also simple. If the number in question is too big then $\frac{x_1}{\sqrt{N}}$ will be too big. Therefore for given r_1 min can be attained at a point which is closer to r_1.

The values of $q_{s_1}/\sqrt{N}$ and $h_{s_1},h_{s_1+1}\dots,\ h_{s_1+m_2}$ determine the structure of the partitions $\eta^{(s_1)},\dots,\eta^{(s_1+m_2)}$.

The conclusion which follows from both lemmas is that for each r_1 we check only finitely many x_1 and x_2 and find $\min(x_1\alpha_1+x_2\alpha_2)$ among them. The number of points which have to be checked depends on $\alpha_1,\alpha_2,\ \frac{q_{s_1}}{\sqrt{N}}$ and $h_{s_1},\dots,h_{s_1+m_2}$.

Now we remark that r_1 must be also an end-point of some element of the partition $\eta^{(s_1)}$. Indeed, if r_1 increases within some element of $\eta^{(s_1)}$ then the set of values r_1-x_1 which have to be checked remain the same. Then $\max\limits_{r_1}$ is attained at the end-point of this element $\eta^{(s_1)}$ because r_1-x_1 is a monotone increasing function of r_1.

The last step in the proof is the final choice of r_1. As was mentioned above r_1 must be an end-point of some element of $\eta^{(s_1)}$ and $\frac{x_1}{\sqrt{N}}$ takes finitely many values. Therefore r_1 should be chosen so that $x_2/\sqrt{N}$ takes the largest possible value. Take the last point $r_1' = \mathcal{R}^{q_{s_1}-1}0$ on the orbit of 0 of the length q_{s_1}. Assume for definiteness that r_1' lies to the left from 0. Consider m_2 elements of $\eta^{(s_1)}$ which start from r_1' and go left. Then r_1 must be one of the end-points of these elements. Indeed, if r_1 lies more to the left from 0 then the values x_1 take finitely many values and x_2 will be significantly smaller. Therefore it cannot give maximum over r of our basic linear form.

Thus we take m_2 elements of $\eta^{(s_1)}$, consider their end-points. Each end-point is a possible value of r. Taking finitely many x_1 (see Lemma 1 and Lemma 2) we find minimum of our basic linear form. After that we find r for which this minimum takes maximal value. In this way we get the solution of our max-min problem. It is clear that this solution is a function of $\frac{q_{s_1}}{\sqrt{N}}$ and elements $h_j, s_1 \leq j \leq s_1 + m_1$ of the continued fraction of ρ near s_1. Since $\frac{q_{s_1}}{\sqrt{N}}$ and $h_j, s_1 \leq j \leq s_1 + m_1$ have limiting distribution as $N \to \infty$ the number $f_N(a) = \frac{1}{N^{3/2}} F_1(a)$ also has a limiting distribution.

It remains to extend our proof to the case when the pairs from a_1, a_2, a_3 have non-trivial common divisors, say k_1 is *gcd* of a_1, a_3 and k_2 is *gcd* of a_2, a_3. It is easy to show that k_1, k_2 have a joint limiting probability distribution in the whole ensemble Q_N. Fixing k_1, k_2 we can write $a_1 = k_1 a_1'$, $a_2 = k_2 a_2'$, $a_3 = k_1 k_2 a_3'$ where a_1', a_3' are coprime, a_2', a_3 are coprime and k_1, k_2 are coprime. This implies that $(a_1')^{-1} (\bmod a_3')$ exists and we can multiply both sides of (3) by $(a_1')^{-1}$. This will give

$$k_1 x_1 + k_2 a_2' \cdot (a_1')^{-1} \cdot x_2 \equiv r_1 (\bmod a_3) \tag{9}$$

where $r_1 = r \cdot (a_1')^{-1} \ (\bmod a_3)$. Denote $b = a_2'(a_1')^{-1}$.

Then from (9) we have the linear form

$$k_1 x_1 + k_2 \, b x_2 \equiv r_1 \, (\bmod a_3) \tag{10}$$

which we can treat in the same way as before.

3 Statistical properties of continued fractions

Statistical properties of elements of continued fractions usually are identical for real numbers and for rationales with bounded denominators (see [8]–[9]).

Let $\mathcal{M}$ be the set of integer matrices $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right)$ with determinant $\det S = \pm 1$ such that $1 \leqslant Q \leqslant Q'$, $0 \leqslant P \leqslant Q$, $1 \leqslant P' \leqslant Q'$. For real $\alpha \in (0,1)$ the fractions P/Q and P'/Q' with $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$ will be consecutive convergents to α (distinct from α) if and only if

$$0 < \frac{Q'\alpha - P'}{-Q\alpha + P} = S^{-1}(\alpha) < 1$$

(see [8, lemma 1]). Moreover if $\alpha = [0; h_1, h_2, \ldots]$ then for some $s \geqslant 1$

$$\frac{P}{Q} = [0; h_1, \ldots, h_{s-1}], \quad \frac{P'}{Q'} = [0; h_1, \ldots, h_s], \qquad (11)$$
$$\frac{Q}{Q'} = [0; h_s, \ldots, h_1], \quad \frac{Q'\alpha - P'}{-Q\alpha + P} = [0; h_{s+1}, h_{s+2}, \ldots].$$

It means that the distribution of partial quotients $h_{s-k}, \ldots, h_{s+k}$ depends on Gauss-Kuz'min statistics of fractions Q/Q' and $(Q'\alpha - P')/(-Q\alpha + P)$.

For real α, x_1, x_2, y_1, $y_2 \in (0,1)$ denote by $N_{x_1,x_2,y_1,y_2}(\alpha, R)$ the number of solutions of the following system of inequalities

$$0 < S^{-1}(\alpha) \leqslant x_1, \quad Q \leqslant x_2 Q', \quad Q \leqslant y_1 R, \quad R \leqslant y_2 Q', \qquad (12)$$

with variables P, P', Q, Q' such that $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$. Let

$$N(R) = N_{x_1,x_2,y_1,y_2}(R) = \int_0^1 N_{x_1,x_2,y_1,y_2}(\alpha, R)\, d\alpha$$

and

$$G(x_1, x_2, y_1, y_2) = \begin{cases} \frac{2}{\zeta(2)} \left(\log(1 + x_1 x_2) \log \frac{y_1 y_2}{x_2} - \mathrm{Li}_2(-x_1 x_2) \right), & \text{if } x_2 \leqslant y_1 y_2; \\ -\frac{2}{\zeta(2)} \mathrm{Li}_2(-x_1 y_1 y_2), & \text{if } x_2 > y_1 y_2, \end{cases}$$

where $\mathrm{Li}_2(\cdot)$ is the dilogarithm

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\log(1-t)}{t} dt.$$

The next statement implies Theorem 2.

Proposition 1. *For $R \geqslant 2$*

$$N(R) = G(x_1, x_2, y_1, y_2) + O\left(\frac{x_1 \log R}{R} \right).$$

Proof. For every number $\alpha = [0; a_1, a_2, \ldots]$ find a unique matrix $S \in \mathcal{M}$ with elements P, P', Q, Q' defined by (11) with the additional restriction $Q \leqslant R < Q'$. The inequalities $0 < S^{-1}(\alpha) \leqslant x_1$ define the interval $I_{x_1}(S) \subset (0,1)$ of the length

$$|I_{x_1}(S)| = \left| \frac{P' + x_1 P}{Q' + x_1 Q} - \frac{P'}{Q'} \right| = \frac{x_1}{Q'(Q' + x_1 Q)}.$$

Hence

$$N(R) = \sum_{\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}} [Q \leqslant x_2 Q', Q \leqslant y_1 R, R \leqslant y_2 Q'] \frac{x_1}{Q'(Q' + x_1 Q)},$$

where $[A]$ is 1 if the statement A is true, and it is 0 otherwise. Second row (Q, Q') can be complemented to the matrix from $\mathcal{M}$ in two ways. That is why

$$N(R) = 2 \sum_{Q' \geqslant R/y_2} \sum_{(Q,Q')=1} [Q \leqslant x_2 Q', Q \leqslant y_1 R] \frac{x_1}{Q'(Q' + x_1 Q)}. \tag{13}$$

In the first case $x_2 \leqslant y_1 y_2$ and the Möbius inversion formula gives

$$\begin{aligned}
N(R) =& 2 \sum_{d \leqslant R} \frac{\mu(d)}{d^2} \sum_{R/(y_2 d) \leqslant Q' < y_1 R/(x_2 d)} \sum_{Q \leqslant x_2 Q'} \frac{x_1}{Q'(Q' + x_1 Q)} + \\
&+ 2 \sum_{d \leqslant R} \frac{\mu(d)}{d^2} \sum_{Q' \geqslant y_1 R/(x_2 d)} \sum_{Q \leqslant y_1 R/d} \frac{x_1}{Q'(Q' + x_1 Q)} = \\
=& \frac{2}{\zeta(2)} \left(\log(1 + x_1 x_2) \log \frac{y_1 y_2}{x_2} + \int_{1/(x_1 x_2)}^{\infty} \log\left(1 + \frac{1}{t}\right) \frac{dt}{t} \right) + O\left(\frac{x_1 \log R}{R} \right) = \\
=& \frac{2}{\zeta(2)} \left(\log(1 + x_1 x_2) \log \frac{y_1 y_2}{x_2} - \mathrm{Li}_2(-x_1 x_2) \right) + O\left(\frac{x_1 \log R}{R} \right).
\end{aligned}$$

The second case $x_2 > y_1 y_2$ can be treated in the same way. $\square$

Let

$$L(R) = L_{x_1,x_2,y_1,y_2}(R) = \sum_{b \leqslant R^2} \sum_{\substack{a \leqslant b \\ (a,b)=1}} N_{x_1,x_2,y_1,y_2}\left(\frac{a}{b}, R \right).$$

Theorem 3 will be proved in the following form.

Proposition 2. *For $R \geqslant 2$*

$$\frac{2\zeta(2)}{R^4} L(R) = G(x_1, x_2, y_1, y_2) + O\left(\frac{x_1 \log^2 R}{R} \right).$$

Proof. Let $\alpha = a/b$ be a given number and $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$ be a solution of the system (12). Denote by m and n the integers such that $mP + nP' = a, mQ + nQ' = b$. Then the system (12) can be written as follows

$$\begin{gathered}
mP + nP' = a, \quad mQ + nQ' = b, \\
0 < m/n \leqslant x_1, \quad 0 < Q/Q' \leqslant x_2, \quad Q \leqslant y_1 R, \quad R \leqslant y_2 Q'.
\end{gathered}$$

Summing up solutions of this system over a and b we get that the sum $L(R)$ equals to the number of solutions of the following system

$$mQ + nQ' \leqslant R^2, \quad 0 < m/n \leqslant x_1, \quad 0 < Q/Q' \leqslant x_2, \quad Q/y_1 \leqslant R < y_2 Q',$$

where $\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$, $0 \leqslant m \leqslant n$, $(m,n)=1$. For given Q and Q' values of P and P' can be founded in two ways. Number of solutions of the last system is equal to the area of the corresponding region with the factor $1/\zeta(2)$ (see [12, Ch. II, problems 21–22])

$$\frac{R^4}{2\zeta(2)} \cdot \frac{x_1}{Q'(Q'+x_1Q)} + O\left(\frac{x_1R^2\log R}{Q'}\right).$$

It leads to the sum similar to (13):

$$L(R) = \frac{R^4}{\zeta(2)} \sum_{R/y_2 \leqslant Q' \leqslant R^2} \sum_{\substack{Q \leqslant \min\{y_1R, x_2Q'\} \\ (Q,Q')=1}} \frac{x_1}{Q'(Q'+x_1Q)} + O(x_1R^3\log^2 R).$$

Therefore

$$L(R) = \frac{R^4}{\zeta(2)} N(R) + O(x_1R^3\log^2 R),$$

and Proposition 2 follows from Proposition 1. $\square$

In order to prove theorem 4 we have to use Kloosterman sums

$$K_q(m,n) = \sum_{x,y=1}^{q} \delta_q(xy-1)\, e^{2\pi i \frac{mx+ny}{q}},$$

where $\delta_q(a)$ is characteristic function of divisibility by q:

$$\delta_q(a) = [a \equiv 0 \pmod{p}] = \begin{cases} 1, & \text{if } a \equiv 0 \pmod{q}, \\ 0, & \text{if } a \not\equiv 0 \pmod{q}. \end{cases}$$

Using Estermann bound (see [2])

$$|K_q(m,n)| \leqslant \sigma_0(q) \cdot (m,n,q)^{1/2} \cdot q^{1/2}.$$

it is easy to prove the following statement (see [9] for details).

Lemma 3. *Let $q \geqslant 1$ be an integer, Q_1, Q_2, P_1, P_2 be real numbers and $0 \leqslant P_1, P_2 \leqslant q$. Then the sum*

$$\Phi_q(Q_1,Q_2;P_1,P_2) = \sum_{\substack{Q_1 < u \leqslant Q_1+P_1 \\ Q_2 < v \leqslant Q_2+P_2}} \delta_q(uv-1)$$

satisfies the asymptotic formula

$$\Phi_q(Q_1,Q_2;P_1,P_2) = \frac{\varphi(q)}{q^2} \cdot P_1P_2 + O\left(\psi(q)\right),$$

where

$$\psi(q) = \sigma_0(q)\log^2(q+1)q^{1/2}.$$

It implies the following general result (see [8]).

Lemma 4. *Let $q \geqslant 1$ be an integer and let $a(u,v)$ be a function defined on the set of integral points (u,v) such that $1 \leqslant u, v \leqslant q$. Assume that this function satisfies the inequalities*

$$a(u,v) \geqslant 0, \quad \Delta_{1,0} a(u,v) \leqslant 0, \quad \Delta_{0,1} a(u,v) \leqslant 0, \quad \Delta_{1,1} a(u,v) \geqslant 0 \tag{14}$$

at all points at which these conditions have the well-defined meaning. Then the sum

$$W = \sum_{u,v=1}^{q} \delta_q(uv-1) a(u,v)$$

satisfies the asymptotics

$$W = \frac{\varphi(q)}{q^2} \sum_{u,v=1}^{q} a(u,v) + O\left(A\psi(q)\sqrt{q}\right),$$

where $\psi(q)$ is the function from lemma 3 and $A = a(1,1)$ is the maximum of the function $a(u,v)$.

Let

$$\begin{aligned} N_z(R) =& N_{z,x_1,x_2,y_1,y_2}(R) = \int_0^z N_{x_1,x_2,y_1,y_2}(\alpha, R)\, d\alpha, \\ L_z(R) =& L_{z,x_1,x_2,y_1,y_2}(R) = \sum_{b \leqslant R^2} \sum_{\substack{a \leqslant zb \\ (a,b)=1}} N_{x_1,x_2,y_1,y_2}\left(\frac{a}{b}, R\right). \end{aligned}$$

The next statement implies Theorem 4.

Proposition 3. *For $R \geqslant 2$*

$$\begin{aligned} N_z(R) =& z \cdot G(x_1,x_2,y_1,y_2) + O\left(\frac{x_1 \log^3 R}{R^{1/2}}\right), \\ \frac{2\zeta(2)}{R^4} L_z(R) =& z \cdot G(x_1,x_2,y_1,y_2) + O\left(\frac{x_1 \log^3 R}{R^{1/2}}\right). \end{aligned}$$

Proof. Let

$$\mathcal{M}_z = \left\{ \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} \in \mathcal{M} : \frac{P'}{Q'} \leqslant z \right\}.$$

For a given z there is at most one matrix $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$ such that $Q \leqslant R < Q'$ and $z \in I_{x_1}(S)$. Hence

$$N_z(R) = \sum_{\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}_z} [Q \leqslant x_2 Q', Q \leqslant y_1 R, R \leqslant y_2 Q'] \frac{x_1}{Q'(Q'+x_1 Q)} + O\left(\frac{x_1}{R^2}\right).$$

If Q' is fixed then P' and Q satisfy the congruence $P'Q \equiv \pm 1 \pmod{Q'}$. Therefore

$$N_z(R) = \sum_{Q' \geqslant R/y_2} \sum_{P',Q=1}^{Q'} \delta_{Q'}(P'Q \pm 1)[Q \leqslant \min\{x_2 Q', y_1 R\}, P' \leqslant zQ'] \frac{x_1}{Q'(Q'+x_1 Q)} + O\left(\frac{x_1}{R^2}\right).$$

Using Lemma 4 we obtain

$$N_z(R) = \sum_{Q' \geqslant R/y_2} \frac{\varphi(Q')}{(Q')^2} \sum_{P',Q=1}^{Q'} [Q \leqslant \min\{x_2 Q', y_1 R\}, P' \leqslant zQ'] \frac{x_1}{Q'(Q'+x_1 Q)} + O\left(\frac{x_1 \log^3 R}{R^{1/2}}\right) =$$

$$= z \sum_{Q' \geqslant R/y_2} \frac{\varphi(Q')}{Q'} \sum_{Q=1}^{Q'} [Q \leqslant \min\{x_2 Q', y_1 R\}] \frac{x_1}{Q'(Q'+x_1 Q)} + O\left(\frac{x_1 \log^3 R}{R^{1/2}}\right).$$

Applying the formula

$$\frac{\varphi(Q')}{Q'} = \sum_{d|Q'} \frac{\mu(d)}{d} \tag{15}$$

we get the same sum as in the proof of Proposition 1.

As in Proposition 2 the sum $L_z(R)$ equals to the number of solutions of the system

$$mQ + nQ' \leqslant R^2, \quad mP + nP' \leqslant z(mQ + nQ'),$$
$$0 < m/n \leqslant x_1, \quad 0 < Q/Q' \leqslant x_2, \quad Q/y_1 \leqslant R < y_2 Q',$$

where $\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$, $0 \leqslant m \leqslant n$, $(m,n) = 1$. Again, there is at most one matrix $S = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}$ such that $Q \leqslant R < Q'$ and $z \in I_{x_1}(S)$. Also for $Q' \geqslant R$

$$\sum_{n \geqslant 1} \sum_{m \leqslant x_1 n} [mQ + nQ' \leqslant R^2] \ll x_1 R^2.$$

This estimate implies that

$$L_z(R) = \frac{R^4}{\zeta(2)} \sum_{\left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{M}_z} [R/y_2 \leqslant Q' \leqslant R^2, Q \leqslant \min\{y_1 R, x_2 Q'\}] \frac{x_1}{Q'(Q'+x_1 Q)} + O(x_1 R^3 \log^2 R) =$$

$$= \frac{R^4}{\zeta(2)} \sum_{R/y_2 \leqslant Q' \leqslant R^2} \sum_{P',Q=1}^{Q'} [Q \leqslant \min\{y_1 R, x_2 Q'\}, P' \leqslant zQ'] \frac{x_1 \delta_{Q'}(P'Q \pm 1)}{Q'(Q'+x_1 Q)} + O(x_1 R^3 \log^2 R).$$

Using Lemma 4 one more time we obtain

$$L_z(R) = \frac{R^4}{\zeta(2)} \sum_{Q' \geqslant R/y_2} \frac{\varphi(Q')}{(Q')^2} \sum_{P',Q=1}^{Q'} [Q \leqslant \min\{x_2 Q', y_1 R\}, P' \leqslant zQ'] \frac{x_1}{Q'(Q'+x_1 Q)} + O\left(x_1 R^{7/2} \log^3 R\right) =$$

$$= \frac{zR^4}{\zeta(2)} \sum_{Q' \geqslant R/y_2} \frac{\varphi(Q')}{Q'} \sum_{Q=1}^{Q'} [Q \leqslant \min\{x_2 Q', y_1 R\}] \frac{x_1}{Q'(Q'+x_1 Q)} + O\left(x_1 R^{7/2} \log^3 R\right).$$

Applying formula (15) we get the same sum as in as in the proof of Proposition 1. □

Remark 1. *In the simplest case $x_2 = y_1 = y_2 = 1$ we have cumulative distribution function*

$$F(x) = F(x,1,1,1) = -\frac{2}{\zeta(2)}\mathrm{Li}_2(-x),$$

which is not equal to the Gaussian function $\log_2(1+x)$*. As $x \to 0$ function $F(x)$ (with error terms in Propositions 1 and 2) decreases as a linear function $F(x) \sim 2x/\zeta(2)$. This fact shows that the expectation of the partial quotient a_s (defined by inequalities $q_{s-1} \leqslant R < q_s$) is equal to infinity.*

4 Concluding remarks

The calculations done by one of the authors (A. Ustinov) shows that the density of the limiting distribution of $\frac{F(a_1,a_2,a_3}{\sqrt{a_1a_2a_3}}$ has the following simple form:

$$p(t) = \begin{cases} 0, & \text{if } t \in [0,\sqrt{3}]; \\ \frac{12}{\pi}\left(\frac{t}{\sqrt{3}} - \sqrt{4-t^2}\right), & \text{if } t \in [\sqrt{3},2]; \\ \frac{12}{\pi^2}\left(t\sqrt{3}\arccos\frac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}} + \frac{3}{2}\sqrt{t^2-4}\log\frac{t^2-4}{t^2-3}\right), & \text{if } t \in [2,+\infty). \end{cases}$$

This result will be published elsewhere.

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